Title: **Semilinear spaces and matrices over semirings**

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Registration No:**VB-569 OF 2011-2012**;

Date of Registration: **Dec 12, 2013**

Key words: semirings; matrices, semilinear space; graphs; inner product; rank function; linear operator; spectrums.

Linear algebra over distributive lattices, Boolean algebras, incline algebras and semirings etc. is one of the central topics in the modern theory of linear algebra due to their applications to computer science, optimizations and theoretical physics [4, 7]. Several authors studied invertibility of matrices and linear operators on the vector spaces over such algebras [3, 6, 10, 11, 12, 14, 15, 16], whereas some others are developing the theory of vector spaces over such algebras [3, 4, 5, 13].

Semirings are among the most numerous objects in mathematics. Indeed, the first mathematical structure we encounter-the set of all natural numbers-is a semiring. Other semirings arise naturally in such diverse areas of mathematics as combinatorics, functional analysis, topology, graph theory, Euclidean geometry, probability theory, commutative and noncommutative ring theory, optimization theory, discrete event dynamical systems, automata theory, formal language theory and the mathematical modeling of quantum physics and parallel computation systems.

A *semiring* $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the *additive reduct* $(S, +)$ and the *multiplicative reduct* $(S, \cdot)$ are semigroups and such that the following distributive laws hold:

$$x(y + z) = xy + xz \text{ and } (x + y)z = xz + yz.$$

In the theory of matrices over semirings, an invertible matrix is an important type of matrices. In 1952, Luce [8] discussed Boolean matrices over two elements Boolean algebra which is a special kind of semirings, and showed that a matrix possesses a two-sided inverse if and only if it is an orthogonal matrix. Since then, the theory of invertible matrices over semirings has been extensively investigated. For instance, Rutherford [15] obtained necessary and sufficient conditions for the existence of the inverses of Boolean matrices. Giveon developed the theory of invertible lattice matrices over an arbitrary distributive lattice with 0 and 1 which is also a special kind of semirings, thus generalizing Luces discussion on invertible Boolean matrices. Reutenauer and Straubing [14] discussed the inversion of matrices over a commutative semiring. Han and Li [6] gave some necessary and sufficient conditions for an incline matrix to be invertible and presented Cramers ruler over inclines. Tan [16] considered the invertible matrices over general commutative zerosumfree semirings (a commutative zerosumfree semiring is called an antiring in [16]), obtained some necessary and sufficient conditions for a matrix over a commutative antiring to be invertible and showed Cramers ruler, therefore generalized the corresponding results in the literature for Boolean matrices, fuzzy matrices, lattice

matrices and incline matrices. Recently, Dolžan and Oblak [2] characterized invertible matrices and the structure of the set of all invertible matrices over antirings. On the other hand, significant effort has been devoted to building a theory of semilinear spaces similar to that of linear algebra to systems of linear equations, eigenvalue problems, linear independence and rank over various kinds of semirings [1], [5]. In 2009, Gudder and Latremoliere [5] developed inner product spaces over a Boolean algebra. Gudder and Latremoliere generalized some results in [9] on power of matrices, established for the Boolean algebra $\{0, 1\}$ to arbitrary Boolean algebras.

In the proposed thesis we would like to develop the theory of linear independence, orthogonality etc. in semilinear spaces over a semiring with a special emphasis to the semirings whose additive reduct is a semilattice and matrices over these semirings.

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